LECTURE NO 7

Curl of a vector

The **curl** of **A** is an axial (or rotational) vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.[2]

$$\text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right)_{\max} \mathbf{a}_n$$

Divergence of Vector

The divergence of **A** at a given point *P* is the *outward* flux per unit volume as the volume shrinks about *P*.

$$\text{div}\,\mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

We can obtain an expression for $\nabla \cdot \mathbf{A}$ in Cartesian coordinates from the defini eq. (3.32). Suppose we wish to evaluate the divergence of a vector field $\mathbf{A}$ a $P(x_o, y_o, z_o)$; we let the point be enclosed by a differential volume as in Figure 3.1 surface integral in eq. (3.32) is obtained from

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \left(\int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}} \right) \mathbf{A} \cdot d\mathbf{S}$$

A three-dimensional Taylor series expansion of A_x about P is

$$A_x(x, y, z) = A_x(x_o, y_o, z_o) + (x - x_o) \frac{\partial A_x}{\partial x}\bigg|_P + (y - y_o) \frac{\partial A_x}{\partial y}\bigg|_P + (z - z_o) \frac{\partial A_x}{\partial z}\bigg|_P + \text{higher-order terms}$$

For the front side, $x = x_o + dx/2$ and $d\mathbf{S} = dy\,dz\,\mathbf{a}_x$. Then,

$$\int_{\text{front}} \mathbf{A} \cdot d\mathbf{S} = dy\,dz \left[A_x(x_o, y_o, z_o) + \frac{dx}{2} \frac{\partial A_x}{\partial x}\bigg|_P \right] + \text{higher-order terms}$$

For the back side, $x = x_o - dx/2$, $d\mathbf{S} = dy\,dz(-\mathbf{a}_x)$. Then,

$$\int_{\text{back}} \mathbf{A} \cdot d\mathbf{S} = -dy\,dz \left[A_x(x_o, y_o, z_o) - \frac{dx}{2} \frac{\partial A_x}{\partial x}\bigg|_P \right] + \text{higher-order terms}$$

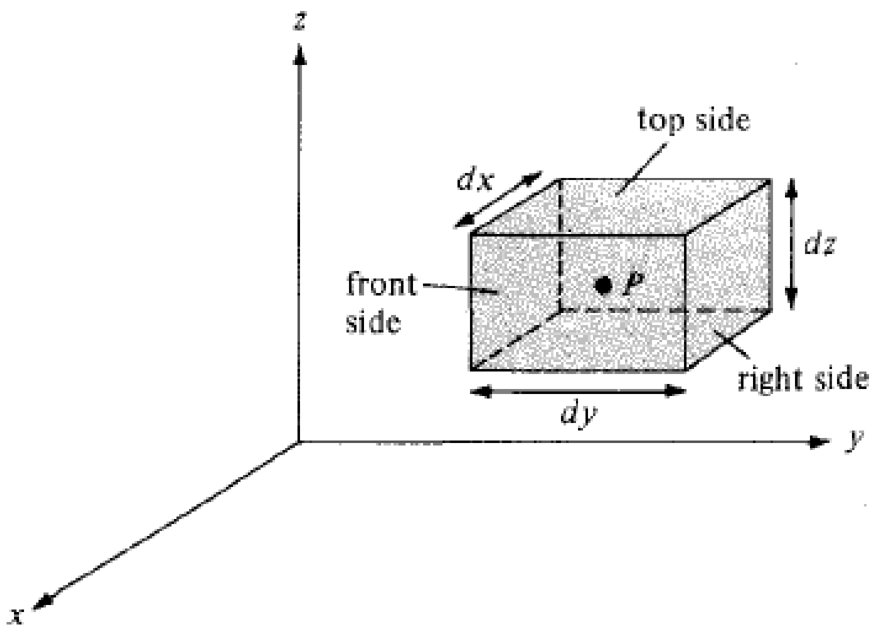

Figure 3.15 Evaluation of $\nabla \cdot \mathbf{A}$ at point $P(x_0, y_0, z_0)$.

Hence,

$$\int_{\text{front}} \mathbf{A} \cdot d\mathbf{S} + \int_{\text{back}} \mathbf{A} \cdot d\mathbf{S} = dx\, dy\, dz \left.\frac{\partial A_x}{\partial x}\right|_P + \text{higher-order terms}$$

By taking similar steps, we obtain

$$\int_{\text{left}} \mathbf{A} \cdot d\mathbf{S} + \int_{\text{right}} \mathbf{A} \cdot d\mathbf{S} = dx\, dy\, dz \left.\frac{\partial A_y}{\partial y}\right|_P + \text{higher-order terms}$$

and

$$\int_{\text{top}} \mathbf{A} \cdot d\mathbf{S} + \int_{\text{bottom}} \mathbf{A} \cdot d\mathbf{S} = dx\, dy\, dz \left.\frac{\partial A_z}{\partial z}\right|_P + \text{higher-order terms}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$